

Heat and Mass transfer

Transient Conduction

Transient Conduction (Lumped Capacitance)

Consider a hot metal, initially at a temp T_0 , quenched by immersing in a liquid at T_∞ . [$T_\infty < T_0$]. The essence of lumped capacitance method is the assumption that the temperature of the solid is spatially uniform at any instant during the transient process.

Rate of change of internal Energy of the solid = Rate of heat loss at the surface of the solid

$$\rho c_p \forall \frac{dT}{dt} = -hA(T - T_\infty)$$

[This is like capacitance (metal) in series with resistance (fluid film)]

$$\text{or, } \frac{dT}{dt} = \frac{-hA}{\rho c_p \forall} (T - T_\infty)$$

$$\text{or, } \frac{d\theta}{dt} = \frac{-hA}{\rho c_p \forall} \theta$$

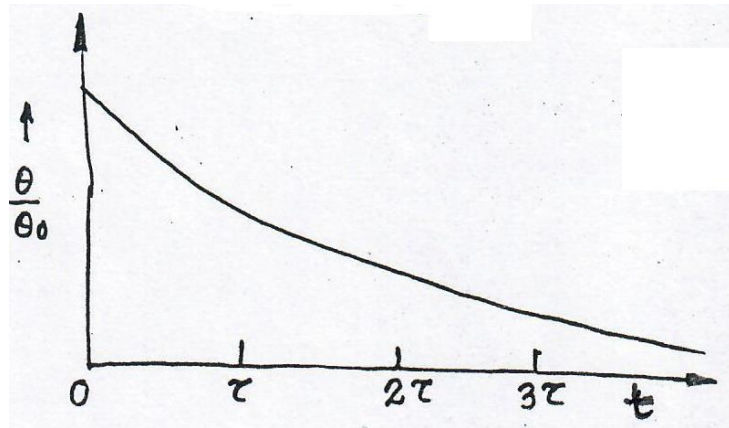
$$\text{or, } \frac{d\theta}{\theta} = -\frac{dt}{\tau} \quad \left[\text{where } \tau \text{ is time constant} = \frac{\rho c_p \forall}{hA} \right]$$

$$\text{or, } \ln \theta = -\frac{t}{\tau} + C_1 \quad \left| \underline{BC} \right. \text{ at } t = 0, \theta = \theta_0$$

$$\text{applying the } BC \Rightarrow C_1 = \ln \theta_0$$

$$\text{or, } \ln \left(\frac{\theta}{\theta_0} \right) = -t/\tau \quad \text{or} \quad \theta = \theta_0 e^{-t/\tau}$$

Time constant τ gives us the idea as to how the ambient temperature is reached.



Concept of Biot number

$$Bi = \frac{hL}{k_s} = \frac{hLA}{k_s A} = \frac{L/k_s}{1/hA}$$

$$= \frac{\text{Conductive resistance}}{\text{Fluid film resistance}}$$

The aforesaid transient analysis is valid for $Bi \ll 1$. The temperature gradient within the solid is small and $T(x,t) = T(t)$. The temperature difference is between the solid and fluid.

If the following condition is satisfied, the error associated with the lumped capacitance method is small:

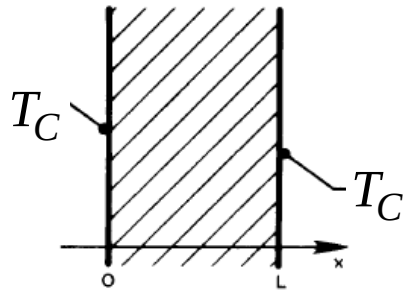
$$Bi \ll 0.1$$

From the equation $\theta = \theta_0 e^{-t/\tau}$ we can write that the exponent is $\frac{hA t}{\rho c_p \forall}$

$$\text{now, } \frac{hAt}{\rho c_p \forall} = \frac{hL}{k_s} \cdot \frac{k_s}{\rho c_p} \cdot \frac{t}{L^2} = Bi \frac{\alpha t}{L^2} = Bi Fo$$

where $Fo = \frac{\alpha t}{L^2}$ is Fourier number. It is nothing but nondimensional time. $\theta = \theta_0 e^{-Bi Fo}$

Analysis of One-dimensional transient conduction in a 1-D slab



Governing equation $\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$

at $x = 0$ and $x = L$, $T = T_C$, for $t > 0$

at $t = 0$, $T = T_0$, for $0 \leq x \leq L$

$\theta = (T - T_C)$; $\theta_0 = (T_0 - T_C)$

We assume solution, $\theta = T(x, t) = X(x)\Gamma(t)$

$$\frac{\partial \theta}{\partial t} = \frac{\partial T}{\partial t} = X(x)\dot{\Gamma}(t); \quad \frac{\partial^2 T}{\partial x^2} = X''(x)\Gamma(t) = \frac{\partial^2 \theta}{\partial x^2}$$

we find, $\alpha X'' \Gamma = X \dot{\Gamma}$ Governing Eqn. $\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}$

or, $\frac{\dot{\Gamma}}{\alpha \Gamma} = \frac{X''}{X} = \text{const} = \lambda^2 = C$

Left hand side- only function of t ; RHS only function of x

If c is imaginary, $\frac{1}{\alpha \Gamma} \frac{d\Gamma}{dt} = C i$ or $\ln \Gamma = z i t + c_1$

or $\Gamma = \Gamma_0 e^{z i t} \Rightarrow$ we don't want oscillatory solution

If c is positive, $\frac{1}{\alpha \Gamma} \frac{d\Gamma}{dt} = C$ or $\ln \Gamma = \alpha c t + c_1$

or $\Gamma = \Gamma_0 e^{\alpha c t} \Rightarrow$ This is growing in time, we don't want

We consider $\left\{ \frac{1}{\alpha \Gamma} \frac{d\Gamma}{dt} = -\lambda^2 \right.$

$$\text{or, } \frac{d\Gamma}{\Gamma} = -\alpha \lambda^2 dt \quad \text{or, } \ln \Gamma = e^{-\alpha \lambda^2 t} + c'$$

$$\text{or, } \Gamma = c_1 e^{-\alpha \lambda^2 t} \quad \text{is acceptable}$$

$$\text{for R.H.S. } \frac{X''}{X} = -\lambda^2$$

$$X(x) = c_2 \sin \lambda x + c_3 \cos \lambda x$$

$$\text{now } \theta = X(x) \Gamma(t) = e^{-\alpha \lambda^2 t} [A \cos \lambda x + B \sin \lambda x]$$

$$\text{at } x=0 \text{ for } t>0 : \theta=0 \quad (T=T_C)$$

$$\text{Therefore, } A e^{-\alpha \lambda^2 t} = 0 \Rightarrow A=0$$

$$\theta = e^{-\alpha \lambda^2 t} B \sin \lambda x$$

$$\text{The second boundary condition is at } x=L \text{ for } t>0 : \theta=0 \text{ (since } T=T_C)$$

$$0 = e^{-\alpha \lambda^2 t} B \sin \lambda L$$

This is only possible for either $B=0$ or $\sin \lambda L = 0$. If $B=0$, the solution is trivial.

so, $\sin \lambda L = 0$ or $\lambda L = n\pi$, $n=1,2,3 \dots \infty$

$$\text{Therefore, } \theta = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{L}\right)$$

This equation must satisfy the initial condition

$$\theta = F(x) \text{ at } t = 0$$

$$\text{We get, } \theta_0 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Basically this equality is Fourier sine series expansion of θ_0 over the interval $0 \leq x \leq L$ using the properties of orthogonal functions in the interval 0 and L

$$\int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = 0 \text{ if } n \neq m$$

$$\text{If } n = m; \int_0^L \sin^2 \frac{n\pi x}{L} dx = \int_0^L \frac{1}{2} \left[1 - \frac{\cos 2n\pi x}{L} \right] dx = \frac{L}{2}$$

$$\begin{aligned} \text{Similarly } \int_0^L \theta_0 \sin \lambda_n x \, dx &= \int_0^L \left(\sum_{n=1}^{\infty} B_n \sin \lambda_n x \right) \sin \lambda_n x \, dx \\ &= \int_0^L B_n \sin^2 \lambda_n x \, dx = B_n \frac{L}{2} \end{aligned}$$

$$\text{So, } B_n = \frac{2}{L} \int_0^L \theta_0 \sin \lambda_n x \, dx$$

$$\text{or, } B_n = \frac{2\theta_0}{L} \int_0^L \sin \frac{n\pi x}{L} \, dx$$

$$\text{or, } B_n = \frac{2\theta_0}{L} \cdot \frac{L}{n\pi} \left[-\cos \frac{n\pi x}{L} \right]_0^L$$

$$\text{or, } B_n = \frac{2\theta_0}{n\pi} (1 - \cos n\pi)$$

$$\theta = \theta_0 \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - \cos n\pi) \sin \frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}$$

$$B_n = \frac{2\theta_0}{n\pi} (1 - \cos n\pi); \text{ for the even } n, \text{ this term } \Rightarrow 0$$

$$\text{for the odd } n, \text{ this term } = \frac{4\theta_0}{n\pi}$$

$$\text{Therefore, } \theta = \theta_0 \cdot \frac{4}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t} \sin \frac{n\pi x}{L}$$

$$\text{or, } \frac{\theta}{\theta_0} = \frac{4}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}$$

Finally we get

$$\frac{\theta}{\theta_0} = \frac{4}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}$$

Which means the temperature distribution is,

$$\frac{T - T_C}{T_0 - T_C} = \frac{4}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}$$

Here we have found the temperature distribution based on the idea that the temperature at the sides of the slab is known (T_C) after finite time. However, this temperature is dependent on the ambient temperature and the film heat transfer coefficient. We will consider such a situation in the next lecture.

Thank you